A note on a maximum principle in Bénard convection

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It is in this note shown that the generalized entropy production V defined by

$$V = R < \theta u_3 > - \tfrac{1}{2} R^{(0)} < (\nabla u_i)^2 > - \tfrac{1}{2} < (\nabla \theta)^2 >$$

is maximum for the physical realized steady solutions when the inertia terms are small, however finite. The symbols are defined in the introduction of the note.

1. Introduction.

It is well known that when inertia terms are neglected, the motion in thermal convection is governed by a maximum principle (Prigogine 1945). It is thus readily derived that the functional V defined by

$$V = R <\theta u_3> - \tfrac{1}{2}R <(\nabla u_i)^2> - \tfrac{1}{2}<(\nabla\theta)^2> \qquad (1.1)$$

satisfies the condition

$$\frac{\partial V}{\partial t} \geq 0 \qquad (1.2)$$

where the equals sign is only valid when the motion has become steady. Here R is the Rayleigh number, θ the temperature and u_i the velocity. u_3 denotes the vertical component. The symbol $< >$ denotes integration over the entire fluid layer.

From (1.2) we conclude that a steady, stable solution of the problem correponds to a maximum value of V. The second term in (1.1) obviously denotes the dissipation rate whereas the two last terms together constitute the entropy production. The first term expresses the conversion of potential energy, and all three terms together are denoted as the generalized entropy production. (1.2) therefore states that in a stable equilibrium the generalized entropy production obtains a maximum value.

Here the question naturally arises whether such a principle is valid also when the inertia terms are taken into account. This problem shall be discussed in the following sections where we shall show that if the definition of V is slightly modified, (1.2) is also valid for inertia terms different from zero, however assumed to be small.

2. The basic equations and boundary conditions.

We shall here only consider a fluid layer of infinite horisontal extent and bounded by two horizontal boundaries. For simplicity we apply the Boussinesq approximation and disregard any effect due to the material properties being temperature dependent. The equations of motion and continuity may then be written

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho_o} \frac{\partial p}{\partial x_i} - \frac{\rho g}{\rho_o}\delta_{i3} + \nu\nabla^2 u_i \tag{2.1}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2.2}$$

where we have used the summation convention, and i,k may be 1,2,3. x_1, x_2 are horizontal coordinates, x_3 the vertical coordinate measured positive upwards, t the time, u_i the velocity, ρ the density, ρ_o a standard density, p the pressure, g the acceleration of gravity, ν the kinematic viscosity, δ_{ij} the Kronecker delta and ∇^2 is the Laplacian. Furthermore, the equation for conduction of heat is

$$\frac{\partial T}{\partial t} + u_k \frac{\partial T}{\partial x_k} = \kappa\nabla^2 T \tag{2.3}$$

and the equation of state may be written

$$\rho = \rho_o(1-\alpha(T-T_o)) \tag{2.4}$$

Here T denotes the temperature, κ the thermal diffusivity,

α the coefficient of expansion and T_0 is a standard temperature.

The temperature may be written

$$T = T_0 - \beta x_3 + \theta \tag{2.5}$$

where $\beta = \Delta T/h$ with ΔT denoting the difference in temperature between the lower and upper boundary and h the depth of the layer.

To get a dimensionless form of the equations we set

$$x_i = hx_i' , \qquad u_i = \kappa u_i'/h , \qquad t = h^2 t'/\kappa$$

$$\theta = \kappa\nu\theta'/\alpha g h^3 , \; p = \kappa^2 \rho_0 p'/h^2 .$$

Disregarding the static pressure, applying (2.4) and (2.5), and dropping the primes, we obtain

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = - \frac{\partial p}{\partial x_i} + P\theta\delta_{i3} + P\nabla^2 u_i \tag{2.6}$$

$$\frac{\partial \theta}{\partial t} + u_k \frac{\partial \theta}{\partial x_k} = \nabla^2 \theta + Ru_3 \tag{2.7}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2.8}$$

Here P is the Prandtl number and R the Rayleigh number:

$$P = \frac{\nu}{\kappa} \qquad\qquad R = \frac{\alpha g \beta h^4}{\nu\kappa} \tag{2.9}$$

The horizontal boundaries may be either rigid or free. In the first case $u_i = 0$ at the boundary; in the last case the vertical velocity and the shearing stresses are zero at the boundary. It will furthermore be assumed that the horizontal boundaries are either perfect heat conductors or perfect heat insulators. Applying (2.8) we then have

$$u_i = 0, \quad \theta = 0 \quad \text{or} \quad \frac{\partial\theta}{\partial n} = 0 \quad \text{at rigid boundaries}$$

$$u_3 = \frac{\partial^2 u_3}{\partial x_3{}^2} = 0, \quad \theta = 0 \quad \text{or} \quad \frac{\partial\theta}{\partial n} = 0 \quad \text{at free boundaries.}$$

The linearized version of equations (2.6)-(2.8) together with the proper boundary conditions lead to an eigenvalue problem which determine the critical Rayleigh number, $R^{(o)}$, corresponding to the outset of convection. It is easily shown that this problem is selfadjoint. In the linear eigenvalue problem $\partial/\partial t$ operator may therefore be cancelled. Eliminating u_1 and u_2 and applying (2.8) the linearized equations may be written

$$\begin{aligned} &\nabla^4 u_3 + \nabla_2{}^2\theta = 0 \\ &R^{(o)} u_3 + \nabla^2\theta = 0 \end{aligned} \tag{2.10}$$

where $\nabla_2{}^2$ is the two-dimensional Laplacian.

The general solution of (2.10) may be written

$$\begin{aligned} u_3 &= f(x_3)\sum A_n \cos a\,(x_1\cos\phi_n + x_2\sin\phi_n + \psi_n) \\ \theta &= g(x_3)\sum A_n \cos a\,(x_1\cos\phi_n + x_2\sin\phi_n + \psi_n) \end{aligned} \tag{2.11}$$

where $f(x_3)$ and $g(x_3)$ satisfy the differential equations

$$\begin{aligned} &\left(\frac{d^2}{dx_3} - a^2\right)^2 f(x_3) - a^2 g(x_3) = 0 \\ &R^{(o)} f(x_3) + \left(\frac{d^2}{dx_3{}^2} - a^2\right) g(x_3) = 0 \end{aligned} \tag{2.12}$$

3. The maximum principle derived from the amplitude equations.

In order to solve equations (2.6) and (2.7) for small amplitudes, we shall apply the perturbation method. We write

$$R = R^{(o)} + \varepsilon R^{(1)} + \varepsilon^2 R^{(2)} + \dots \tag{3.1}$$

$$u_i = \varepsilon u_i^{(1)} + \varepsilon^2 u_i^{(2)} + \dots \tag{3.2}$$

$$\theta = \varepsilon \theta^{(1)} + \varepsilon^2 \theta^{(2)} + \dots \tag{3.3}$$

$$p = \varepsilon p^{(1)} + \varepsilon^2 p^{(2)} + \dots \tag{3.4}$$

Introducing these expressions in (2.6) and (2.7), we obtain for the first order terms

$$\begin{gathered} - \frac{\partial p^{(1)}}{\partial x_i} + P\theta^{(1)}\delta_{i3} + P\nabla^2 u_i^{(1)} = 0 \\ \nabla^2 \theta^{(1)} + R^{(o)} u_3^{(1)} = 0 \end{gathered} \tag{3.5}$$

For the second order terms we find

$$\begin{gathered} - \frac{\partial p^{(2)}}{\partial x_i} + P\theta^{(2)}\delta_{i3} + P\nabla^2 u_i^{(2)} = u_k^{(1)} \frac{\partial u_i^{(1)}}{\partial x_k} \\ \nabla^2 \theta^{(2)} + R^{(o)} u_3^{(2)} = u_k^{(1)} \frac{\partial \theta^{(1)}}{\partial x_k} \end{gathered} \tag{3.6}$$

In the last equation we have cancelled the term $R^{(1)}u_3$, utilizing that it follows from the existence condition that $R^{(1)}$ is zero. The third order terms fulfil the equations

$$\begin{gathered} - \frac{\partial p^{(3)}}{\partial x_i} + P\theta^{(3)}\delta_{i3} + P\nabla^2 u_i^{(3)} = u_k^{(2)} \frac{\partial u_i^{(1)}}{\partial x_k} + u_k^{(1)} \frac{\partial u_i^{(2)}}{\partial x_k} + \frac{\partial u_i^{(1)}}{\partial t} \\ \nabla^2 \theta^{(3)} + R^{(o)} u_3 = u_k^{(2)} \frac{\partial \theta^{(1)}}{\partial x_k} + u_k^{(1)} \frac{\partial \theta^{(2)}}{\partial x_k} - R^{(2)} u_3^{(1)} + \frac{\partial \theta^{(1)}}{\partial t} \end{gathered} \tag{3.7}$$

For later references we shall write down some forms of the existence conditions for the second and third order equations. Let $u_i^{(1)'}$ and $\theta^{(1)'}$ denote solutions of the first order equation, fulfilling the boundary conditions and (2.8). The existence condition for the second order equation is then obtained by multiplying the first equation in (3.6) with $R^{(o)}u_i^{(1)'}$ and the second equation with $P\theta^{(1)'}$, adding the equations and integrating over the entire fluid layer, applying the boundary conditions. Putting $u_i^{(1)'} = \partial u_i^{(1)}/\partial t$ and $\theta^{(1)'} = \partial\theta^{(1)}/\partial t$, we obtain

$$R^{(o)} < u_k^{(1)} \frac{\partial u_i^{(1)}}{\partial x_k} \frac{\partial u_i^{(1)}}{\partial t} > + P < u_k^{(1)} \frac{\partial \theta^{(1)}}{\partial x_k} \frac{\partial \theta^{(1)}}{\partial t} > = 0 \qquad (3.8)$$

Analogous, by considering (3.6) and (3.7), respectively, and putting $u_i^{(1)'} = u_i^{(1)}$ and $\theta_i^{(1)'} = \theta_i^{(1)}$ we find

$$R^{(o)} <\theta^{(2)}u_3^{(1)}>+R^{(o)} <\theta^{(1)}u_3^{(2)}> - R^{(o)} <\nabla u_i^{(2)}\nabla u_i^{(1)}>$$
$$-<\nabla\theta^{(2)}\nabla\theta^{(1)}> = 0 \qquad (3.9)$$

$$R^{(o)} <\theta^{(3)}u_3^{(1)}> + R^{(o)} <\theta^{(1)}u_3^{(3)}> - R^{(o)} < \nabla u_i^{(3)}\nabla u_i^{(1)}>$$
$$<\nabla\theta^{(3)}\nabla\theta^{(1)}> = 0 \qquad (3.10)$$

From (3.5 - 3.7) we may derive the amplitude equations determining the amplitude $A_n(t)$ in (2.11), provided this is small.

It may be shown that *)

$$K\dot{A}_n = EA_n - \frac{1}{4} A_n \sum B(\lambda) A_m^{\ 2} - \frac{1}{8} B(4) A_n^{\ 3} \tag{3.11}$$

Here

$$K = R^{(o)} \int (f^2 + a^{-2} f'^2) dz + P \int g^2 dz$$

$$E = P \Delta R \int gf dz$$

where

$$\Delta R = R - R^{(o)}.$$

The integration in these formulas takes place between the two horizontal boundaries. Furthermore, an asterisk denotes complex conjugate, and λ is given by

$$\lambda = 4\cos^2 \frac{\phi - \phi'}{2}$$

$B(\lambda)$ is a somewhat more complicated expression and the detailed form of it will not be given here. We note, however, that $B(\lambda)$ is real, and is a function of ϕ and ϕ' through λ only. More explicitly, B is a function of $|\phi' - \phi|$ only. Physically, this is due to the fact that there is no preferred direction in the horizontal plane. Therefore, the effect of a roll in the ϕ-direction on a roll in the ϕ'-direction , is equal to the effect of the roll in the ϕ'-direction on the roll in the ϕ-direction. The problem

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*) The equation in this form is taken from a yet unpublished paper by Torbjørn Ellingsen. The equation implies no phase shift since, for simplicity, the effect of material properties being temperature dependent are here cancelled.

therefore contains a certain form of symmetry. Due to this symmetry, equation (3.11) may be derived from a maximum principle.

Let us introduce the function

$$V = \tfrac{1}{2} E \sum A_n^2 - \tfrac{1}{8} \sum_{m \neq n} B(\lambda) A_n^2 A_m^2 - \tfrac{1}{32} B(4) A_n^4 \tag{3.12}$$

It is then readily seen that

$$K\dot{A}_n = \frac{\partial V}{\partial A_n} \tag{3.13}$$

or, written in vector form,

$$K\dot{\vec{A}} = \nabla V \tag{3.14}$$

where the ∇-operator works in the amplitude space.

It follows from (3.14) that for steady motion $\nabla V = 0$, i.e. the function V has then a stationary value. Furthermore, since K is positive, $d\vec{A}$ is directed towards increasing values of V. Thus we infer that a stable, steady motion corresponds to a maximum value of V.

The existence of such a maximum principle has earlier been derived by Busse (1967) by a somewhat different approach. The intention of this note is, however, to find the physical interpretation of the function V which is maximum for a given Rayleigh number and shows that it is closely connected to the generalized entropy production mentioned in the introduction. To discover this interpretation we, as a first step, multiply (3.13) with $\dot{A}_n$, from which it follows that

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial A_n} \dot{A}_n = K \dot{A}_n^2 \tag{3.15}$$

This shows again that V ever increases until the motion becomes steady. (3.15) shall in the next section be derived directly from the equations of motion from which the physical interpretation of V follows immediately.

4. Direct derivation of the maximum principle.

We multiply (2.6) with $R\frac{\partial u_i}{\partial t}$ and (2.7) with $P\frac{\partial \theta}{\partial t}$, add the equations and integrate over the entire fluid layer. Applying the Gaussian theorem and the boundary conditions, we find

$$\frac{\partial}{\partial t} P\left[R<\theta u_3> - \tfrac{1}{2} R <(\nabla u_i)^2> - \tfrac{1}{2}<(\nabla\theta)^2>\right] =$$

$$R(\frac{\partial u_i}{\partial t})^2 + P(\frac{\partial \theta}{\partial t})^2 + R <u_k \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial t}> + P<u_k \frac{\partial \theta}{\partial x_k}\frac{\partial \theta}{\partial t}> \qquad (4.1)$$

To advance further, the two last terms on the right hand side must be given another form. We first note that according to (3.8), the sum of the third order terms on the left hand side vanishes, and according to (3.9) the sum of the third order terms on the right hand side are zero. The two last terms mentioned above are therefore, to the order considered here, of fourth order (or, more precisely, of sixth order since $\partial/\partial t$ is of second order). We may in these terms with sufficient accuracy put $R = R^{(o)}$.

It may be shown that the following fourth order terms are zero (see Schlüter, Lortz and Busse (1965)).

$$<u_k^{(2)}\frac{\partial u_i^{(1)}}{\partial x_k}\frac{\partial u_i^{(1)}}{\partial t}> = <u_k^{(2)}\frac{\partial \theta^{(1)}}{\partial x_k}\frac{\partial \theta^{(1)}}{\partial t}> = 0 \qquad (4.2)$$

We also note that

$$<u_k^{(1)} \frac{\partial\theta^{(2)}}{\partial x_k} \frac{\partial\theta^{(1)}}{\partial t}> = <u_k^{(1)} \frac{\partial\theta^{(1)}}{\partial x_k} \frac{\partial\theta^{(2)}}{\partial t}> - \frac{\partial}{\partial t} <u_k^{(1)} \frac{\partial\theta^{(1)}}{\partial x_k} \theta^{(2)}>$$

$$+ <\frac{\partial u_k^{(1)}}{\partial t} \frac{\partial\theta^{(1)}}{\partial x_k} \theta^{(2)}> \qquad (4.3)$$

Furthermore, applying that the motion is of the form (2.11), it is easily shown that

$$<\frac{\partial u_k^{(1)}}{\partial t} \frac{\partial\theta^{(1)}}{\partial x_k} \theta^{(2)}> = <u_k^{(1)} \frac{\partial^2\theta^{(1)}}{\partial x_k \partial t} \theta^{(2)}> \qquad (4.4)$$

and thereby

$$<\frac{\partial u_k^{(1)}}{\partial t} \frac{\partial\theta^{(1)}}{\partial x_k} \theta^{(2)}> = -<u_k^{(1)} \frac{\partial\theta^{(2)}}{\partial x_k} \frac{\partial\theta^{(1)}}{\partial t}> \qquad (4.5)$$

Thus we obtain that

$$<u_k^{(1)} \frac{\partial\theta^{(2)}}{\partial x_k} \frac{\partial\theta^{(1)}}{\partial t}> + <u_k^{(1)} \frac{\partial\theta^{(1)}}{\partial x_k} \frac{\partial\theta^{(2)}}{\partial t}> =$$

$$\frac{3}{2} <u_k^{(1)} \frac{\partial\theta^{(1)}}{\partial x_k} \frac{\partial\theta^{(2)}}{\partial t}> - \frac{1}{2} \frac{\partial}{\partial t} <u_k^{(1)} \frac{\partial\theta^{(1)}}{\partial x_k} \theta^{(2)}> \qquad (4.6)$$

The terms $<u_k \frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial t}>$ give rise to a quite analogous expression, viz.

$$<u_k^{(1)} \frac{\partial u_i^{(2)}}{\partial x_k} \frac{\partial u_i^{(1)}}{\partial t}> + <u_k^{(1)} \frac{\partial u_i^{(1)}}{\partial x_k} \frac{\partial u_i^{(2)}}{\partial t}> =$$

$$\frac{2}{3} <u_k^{(1)} \frac{u_i^{(1)}}{x_k} \frac{\partial u_i^{(2)}}{\partial t}> - \frac{1}{2} \frac{\partial}{\partial t} <u_k^{(1)} \frac{\partial u_i^{(1)}}{\partial x_k} u_i^{(2)} \qquad (4.7)$$

Returning to equation (3.6), multiplying by

$$R^{(o)} \frac{\partial u_i^{(2)}}{\partial t}, \qquad P \frac{\partial \theta^{(2)}}{\partial t}$$

and

$$R^{(o)} u_i^{(2)}, \qquad P\, \theta^{(2)}$$

and integrating over the entire fluid layer, applying the boundary conditions, we find for the remaining fourth order terms of the right hand side of (4.1)

$$\begin{aligned} & R^{(o)}\left[<u_k^{(1)} \frac{\partial u_i^{(2)}}{\partial x_k} \frac{\partial u_i^{(1)}}{\partial t}> + < u_k^{(1)} \frac{\partial u_i^{(1)}}{\partial x_k} \frac{\partial u_i^{(2)}}{\partial t}>\right] \\ & + P\left[<u_k^{(1)} \frac{\partial \theta^{(2)}}{\partial x_k} \frac{\partial \theta^{(1)}}{\partial t}> + < u_k^{(1)} \frac{\partial \theta^{(1)}}{\partial x_k} \frac{\partial \theta^{(2)}}{\partial t}>\right] = \qquad (4.8) \\ & \tfrac{1}{2} P \frac{\partial}{\partial t}\{R^{(o)}<\theta^{(2)} u_3^{(2)}> - \tfrac{1}{2} R^{(o)}<(\nabla u_i^{(2)})^2> - \tfrac{1}{2}<(\nabla\theta^{(2)})^2>\} \end{aligned}$$

which is in the form of a total time-derivative term.

It is readily seen from (3.5) that

$$<\theta^{(1)} u_3^{(1)}> - <(\nabla u_i^{(1)})^2> = 0 \qquad (4.9$$

Applying that

$$R^{(o)}(<\theta^{(1)} u_3^{(1)}> - \tfrac{1}{2} R^{(o)}<(\nabla u_i^{(1)})^2>) - \tfrac{1}{2}<(\nabla\theta^{(1)})^2> = 0 \qquad (4.10)$$

(4.1) may by means of (4.6) and (4.7) be written

$$\begin{aligned} & \frac{\partial}{\partial t} P\left[R^{(o)} <\theta^{(2)} u_3^{(2)}> - \tfrac{1}{2} R^{(o)} <(\nabla u_i^{(2)})^2>\right. \qquad (4.11) \\ & \left. - \tfrac{1}{2}<(\nabla\theta^{(2)})> + \Delta R <\theta^{(1)} u_3^{(1)}>\right] = 2<R^{(o)}(\frac{\partial u_i}{\partial t})^2 + P(\frac{\partial \theta}{\partial t})^2> \end{aligned}$$

Comparing (4.11) with (3.15) we have thereby shown that, to the order considered, V is given by

$$V = R\,\langle\theta u_3\rangle - \tfrac{1}{2}R^{(o)}\,\langle(\nabla u_i)^2\rangle - \tfrac{1}{2}\,\langle(\nabla\theta)^2\rangle \qquad (4.12)$$

Returning to (1.1) and (1.2) we note that (1.2) is formally still valid in the non-linear regime for small amplitudes if V is slightly redefined as by (4.12). In the linear case (4.12) became identical to (1.1).

We have in the derivation of the maximum principle only considered waves with the same all-over wave number a, say a_o. It is evident that the maximum principle is also true, to the order considered, for waves for which $a = a_o + O(\varepsilon)$.

5. Discussion of the maximum principle. Summary.

It has been shown by Schlüter et al. (1965) that when only waves of the same all-over wave number are considered, just one steady, stable motion exists, namely the two-dimensional roll. Thus the relative maximum of V is in this case an absolute maximum.

For steady motion we find from (2.6) and (2.7) the energy equations

$$\langle\theta u_3\rangle = \langle(\nabla u_i)^2\rangle \qquad (5.1)$$

and

$$\langle(\nabla\theta)^2\rangle = R\,\langle\theta u_3\rangle \qquad (5.2)$$

Addition of these equations gives

$$2R\,\langle\theta u_3\rangle - R\,\langle(\nabla u_i)^2\rangle - \langle(\nabla\theta)^2\rangle = 0 \qquad (5.3)$$

Comparing this equation with (4.12), we find that for steady motion

$$V = \tfrac{1}{2} \Delta R <(\nabla u_i^{(1)})^2> \tag{5.4}$$

We therefore conclude that the physical realized steady solution is characterized by maximum dissipation, implied that the solution is only compared to other possible steady state solutions. Applying (5.1) we may as well say that the convective heat transport is maximum. From (5.2) we likewise deduce that also $<(\nabla\theta)^2>$ is maximum. And, furthermore, we notice that an infinite number of combinations of these quantities are maximum.

The result above that the convective heat transport is a maximum when the various steady motions are compared, was put forward as a hypothesis by Malkus (1954 a,b) and proved to the order considered above by Schlüter et al. (1965). Busse (1967) proved, in agreement with the derivation above, that not only the convective heat transport, but any physical quantity described as an average property of the stationary solution has a maxium value for the physically realized steady motion at a given Rayleigh number. More precisely this means that the form of the cells are characterized by these maximum properties. If we want to determine the realized wave length, we shall find that V has an infinite number of relative maxima (one for every value of a), and it is the initial condition which determines the preferred wave number.

To summarize, we have shown that, to the order considered, the generalized entropy production V defined by (4.12) obtains a maximum when the motion becomes steady. The class of comparable function are those given by (2.11). The underlying maximum principle turns out not to be new, being in fact the same as that found by

Busse (1967). We have, however, here found the explicit physical expression for the function V which is maximum, supposed that the Rayleigh number is given. Thereby we have shown that the (modified) generalized entropy production attains a maximum for steady stable motion also in the non-linear case, provided that the amplitude is small. The derivation of the maximum principle is also somewhat different from that given by Büsse (1965).

If, however, the class of comparable functions are restricted to those corresponding to a steady motion, the physical interpretation of the functional V is not unique.

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